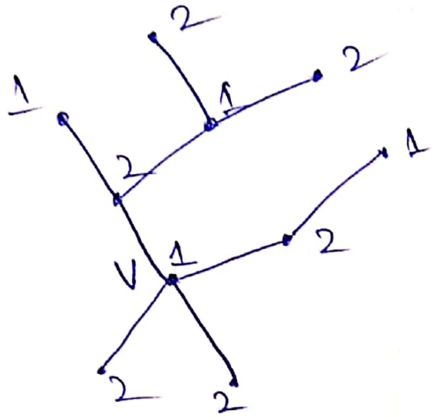
Theorems on Colouring of Graphs

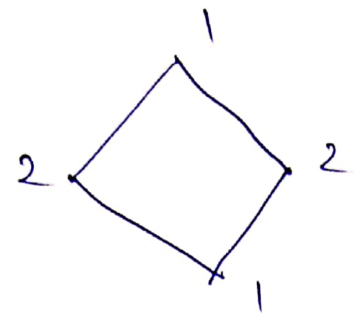
Th 1. Every tree with two or more vertices is 2-chromatic but converse is not true.

Pf:- Let T be a tree with two or more vertices. Choose any vertex 'v' in the T.



Paint v with colour 1. Then paint all the vertices adjacent to v with colour 2. Now paint all the vertices adjacent to those vertices which have been coloured with colour 2, using colour 1. Repeat this process until every vertex of T has been painted as shown in above figure. Since there is one & only one path b/w any two vertices in a tree, no two adjacent vertices will have the same colour. Thus tree T has been properly coloured with two colours. Since T has atleast two vertices and at least one edge, ∴ it can not be coloured with only one colour. ∴ T is 2-chromatic.

Converse, Consider the following graph G →



This graph G is 2-chromatic as shown but it is not a tree as there is circuit in G. ∴ Not every 2-chromatic graph is a tree.

Thm 2) A graph with edges $e \geq 1$ is 2-chromatic iff it has no circuit of odd length.

Pf → Let G be a connected graph with circuits of even lengths only.

To show that G is 2-chromatic i.e. all of its vertices can be coloured using 2 colours only. Select any vertex and paint it with colour 1. Then paint all vertices adjacent to it with colour 2. If a vertex is painted with colour 2 then any vertex adjacent to it will be painted with colour 1. Since every circuit has even length, no adjacent vertices will have the same colour. Hence G is 2-chromatic.

Conversely, if G has a circuit of odd length then we would need at least 3 colours just for that circuit only. ∴ chromatic number of G is ≥ 3, which is a contradiction. ∴ G has no circuit of odd length. (Proved)

Th 31. If Δ is the maximum degree of the vertices in a graph G, then the chromatic number of G is $\leq 1+\Delta$

Pf:- Let G be a graph with 'n' vertices. We shall prove the theorem by induction on the number of vertices. If G has 1 or 2 vertices then the result is clearly true. We assume that the result is true for all graphs having less than n vertices.

Remove any vertex v and all edges incident on v. Then (G-v) is a graph with (n-1) vertices. and maximum degree of any vertex in (G-v) is at most Δ. $\therefore$ by induction, chromatic number of (G-v) $\leq 1+\Delta$

Now we add the vertex 'v' and all edges incident on v to G-v. Since degree of v is Δ, at most Δ colours are needed to colour all vertices adjacent to v. We can assign a different colour to v from $(\Delta+1)$ colours. Hence chromatic number of G $\leq 1+\Delta$.

(Proved)

Thm 4). A graph G of 'n' vertices is a complete graph iff its chromatic polynomial is →

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

Pf → Let G be a complete graph of 'n' vertices $v_1, v_2, \ldots, v_n$. Given λ colours, the first vertex v_1 can be coloured with any of the λ colours, vertex v_2 with any one of remaining $(\lambda-1)$ colours and so on.

$$\therefore P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

Converse, Let

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$ be a chromatic polynomial of a graph G with 'n' vertices. Then by defn of chromatic polynomial, there are λ different ways of colouring any selected vertex. A second vertex can be coloured properly in $(\lambda-1)$ ways, the third in $(\lambda-2)$ ways, ... and the n^{th} in $(\lambda-n+1)$ ways if every vertex is adjacent to every other vertex. But in that case, the graph G is a complete graph.

(Proved)